Full Order LPV/Hinfinity observers for LPV time-delay systems

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FULL ORDER LPV/$\mathcal{H}_\infty$ OBSERVERS FOR LPV TIME-DELAY SYSTEMS

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Abstract: This paper deals about the synthesis of parameter dependent observers for linear parameter varying time delay systems. These observers are computed to satisfy an $\mathcal{H}_\infty$ performance attenuation from the disturbances to the state estimation error. First a stability test is proposed using the Lyapunov-Krasovskii approach and the Newton's formula. It is provided in terms of gridded parameter dependent LMIs. Second, this stability test is derived to obtain a sufficient condition to the existence of a (parameter dependent) observer. Our results are illustrated through an example which demonstrates the benefits of this approach.

Keywords: LPV Time delay systems, Gain-scheduling, observers, LMI

1. INTRODUCTION

Since several years, time-delay systems give rise to more and more attention (see Niculescu (2001), Gu et al. (2003), Fridman (2006), Suplin et al. (2006), Gouaisbaut and Peaucelle (2006)). The delays generally have destabilizing effects and deteriorate performances. Indeed, in the willingness of permanently raising systems performances, the harmful effects of even small delays cannot now be neglected. Moreover, the advent of networks and network controlled systems (NECS) has increased the interest of these kind of systems where appear time-varying delays. For such systems, a dynamical model or measurement of the delays are sometimes considered and used in the controller (see Witrant et al. (2005); Briat et al. (2007)).

During the past decade, LPV systems have shown to allow easy modelling of nonlinear, switched, multimodel systems in a nice fashion. The LPV model is not equivalent to the nonlinear one (see F. Bruzelius and Breitholz (2004)) but using certain methods we can guarantee a sufficient ball radius wherein the vectors field coincides between both models. This assumption is not too restrictive according to the boundedness of control inputs an states from a practical point of view.

Both important facts of LPV systems are the associate LPV control (see Packard (1994), Apkarian and Gahinet (1995), Apkarian and Adams (1998), Scherer (2001)) and LPV observation where measured parameters can be used in order to schedule the controller/observer. This is a very nice alternative to robust control where nonlinearities or time varying parameters were only considered as uncertainties. Now, using the LPV system theory, it is possible to improve performances of the closed-loop system.

While there exist works on LPV time-delay systems stability and control (see Zhang et al. (2002), Wu and Grigoriadis (2001), Zhang and Grigoriadis (2005)), the observer synthesis problem has not been really studied in the literature.

In the present paper, we consider the problem of synthesizing a Luenberger's observer of the form:

$$\begin{aligned}\dot{\hat{x}}(t) &= A(\rho)\hat{x}(t) + A_h\hat{x}_h(t) + B_u(\rho)u(t) \\ &\quad + L(\rho)(y(t) - C(\rho)\hat{x}(t) + C_h(\rho)\hat{x}_h(t))\end{aligned} \quad (1)$$

for multiple time-delay systems of the form:

$$\begin{aligned}\dot{x}(t) &= A(\rho)x(t) + A_h(\rho)x_h(t) + B(\rho)w(t) \\ &\quad + B_u(\rho)u(t) \\ z(t) &= C(\rho)x(t) + C_h(\rho)x_h(t) + D(\rho)w(t) \\ &\quad + D_u(\rho)u(t) \\ y(t) &= C_y(\rho)x(t) + C_{yh}(\rho)x_h(t) + E(\rho)w(t)\end{aligned} \quad (2)$$

where x, x_h, w, z, u and y are respectively the n-dimensional system state, the $d \cdot n$-dimensional delayed state vector defined as $col[x(t-h_i(t))]$, the d-dimensional delay vector $col(h_i(t))$, the m-dimensional exogenous inputs and the q-dimensional controlled outputs, the r-dimensional control inputs and the s-dimensional measured output. The time-varying parameters $\rho(t)$ belong to $\mathscr{P}^k$ and the time-varying delays $h(t)$ to $\mathscr{D}$.

$$\mathscr{P}^k := \left\{\rho \in \mathscr{C}^k(\mathbb{R}^+, U_\rho^k \subset \mathbb{R}^p)\right\},\ k = \{-1, 0, 1\} \quad (3)$$

$$\mathscr{D} := \left\{ h \in \mathscr{C}^1\left(\mathbb{R}^+, \bigcup_i^d [0; \bar{h}_i]\right), |\dot{h}_i(t)| \le \mu_i < 1 \right\} \quad (4)$$

where U_ρ^k is compact, $U_\rho = \bigcup_{-1}^1 U_\rho^k$, $\mathscr{C}^{-1}(I,J)$ denotes the Hilbert-space of discontinuous functions mapping I to J, $\mathscr{C}^0(I,J)$ denotes the Hilbert-space of continuously differentiable functions mapping I to J with discontinuous derivatives and $\mathscr{C}^1(I,J)$ denotes the Hilbert-space of continuously differentiable functions mapping I to J with continuous derivatives.

$\mathscr{P}^{-1}$ denotes the case of discontinuous parameter functions. $\mathscr{P}^0$ considers non-differentiable continuous parameter functions. $\mathscr{P}^1$ describes continuous differentiable parameter functions. Differentiating these cases, will allow us to elaborate more precise sufficient conditions for stability while tackling or not information on parameter derivatives.

In the case when $\rho \in \mathscr{P}^1$, we define next the parameters derivative bounds $\underline{\nu}_i$ and $\bar{\nu}_i$ satisfying $-\infty < \underline{\nu}_i < \dot{\rho}_i < \bar{\nu}_i < \infty$. We introduce the set U_ν such that $\nu \in U_\nu$ and

$$U_\nu := \left\{\nu \in \{\underline{\nu}_1, \bar{\nu}_1\} \times \ldots \times \{\underline{\nu}_p, \bar{\nu}_p\}\right\} \quad (5)$$

The main contribution of the paper is to provide new theorems on full-order observer synthesis for LPV time-delay systems using the Newton's formula without introducing additional dynamics (see Gu et al. (2003) for complements on additional dynamics). The results can be applied to study LTI and LPV time-delay systems stability and are derived to compute a parameter-dependent observer gain. All results are presented for the multiple delay case in the delay dependent framework. We can easily obtain delay-independent criteria while avoiding some matrices or scalars. The validity of the method is demonstrated through an example.

The paper is structured as follows. Section 2 presents the stability/performance lemma providing sufficient condition for time-delay systems stability with an $\mathcal{H}_\infty$ -performance achievement. In section 3, we derive the stability-performance lemma to obtain sufficient conditions for a parameter-dependent observer gain existence and computation feasibility. Section 4 provides discussion on implementation difficulties and their approximate solutions. Section 5 will show that our approach is verified through several examples.

For a family of matrix $\{A_i\}_{i=1,\ldots,n}$, we denote $\mathcal{H}[A_i] = [A_1 \ldots A_n]$ and $\mathcal{V}[A_i] = \mathcal{H}[A_i^T]^T$. For two subspaces I, J. $\mathbb{S}^{n+}$ is the subspace of symmetric positive definite matrices, $\mathbb{R}^n$ is the n-dimensional vector with real components, $\mathbb{R}^{n\times m}$ is the set of matrices with n rows and m columns.

2. STABILITY/PERFORMANCE LEMMA

We provide in this section, a stability lemma based on Newton's formula using the method of He et al. (2004) extended to the parameter dependent case. The result is given in terms of a finite number of infinite dimensional LMIs.

Theorem 2.1. Consider system (2) with parameters $\rho \in \bigcup_{-1}^1 \mathscr{P}^k$ and delays belonging to $\mathscr{D}$. If there exists matrices $Q_i, R_i \in \mathbb{S}^{n+}$, a continuously differentiable matrix valued function $P : U_\rho^1 \rightarrow \mathbb{S}^{n+}$, a scalar $\gamma > 0$ and d^2+3d+2 continuous matrix functions $T_i, L_i, S, M_{ij}, U, N_i : U_\rho \rightarrow \mathbb{R}^{n\times n}$, $E_i : U_\rho \rightarrow \mathbb{R}^{m\times n}$, $i,j = 1,\ldots,d$ such that the 2^p infinite dimensional LMIs (6) hold for all $\nu \in U_\nu$ and for all $\rho \in U_\rho$.

$$\Lambda(\nu,\rho) < 0 \quad (6)$$

where $\Lambda = [\Lambda_{ij}]_{i,j}$, $j = 1\ldots,6$, $\Lambda = \Lambda^T$ and

$$\begin{aligned}\Lambda_{11} &= \sum_i^d (Q_i + Sym(L_i(\rho) - S(\rho)A(\rho)/d)) \\ &\quad + \nu\frac{\partial P}{\partial \rho} \\ \Lambda_{21} &= -T(\rho)A(\rho) + \tilde{M}(\rho) - L(\rho)^T - A_h(\rho)^T S(\rho)^T \\ \Lambda_{22} &= -Q_\mu - Sym(M(\rho) + T(\rho)A_h(\rho)) \\ \Lambda_{31} &= P(\rho) + \sum_i^d N_i(\rho) - U(\rho)A(\rho) + S(\rho)^T\end{aligned}$$

$$
\begin{aligned}
\Lambda_{32} &= -N(\rho) - U(\rho)A_h(\rho) + T(\rho)^T \\
\Lambda_{33} &= \sum_i^d \bar{h}_i R_i + Sym(U(\rho)) \\
\Lambda_{41} &= \sum_i^d E_i(\rho) - B(\rho)^T S(\rho)^T \\
\Lambda_{42} &= -E(\rho) - B(\rho)^T T(\rho)^T
\end{aligned}
$$

$$
\begin{aligned}
\Lambda_{43} &= B(\rho)^T U(\rho)^T & \Lambda_{44} &= -\gamma I_m \\
\Lambda_{51} &= C(\rho) & \Lambda_{52} &= C_h(\rho) \\
\Lambda_{53} &= 0_{q\times n} & \Lambda_{54} &= D(\rho) \\
\Lambda_{55} &= -\gamma I_q & \Lambda_{61} &= \bar{h}L(\rho)^T \\
\Lambda_{62} &= \bar{h}\bar{M}(\rho)^T & \Lambda_{63} &= \bar{h}N(\rho)^T \\
\Lambda_{64} &= \bar{h}E(\rho)^T & \Lambda_{65} &= 0_{n\cdot d\times q} \\
\Lambda_{66} &= -blockdiag_i^d[\bar{h}_i R_i]
\end{aligned}
$$

and $T = \mathcal{V}_i^d[T_i]$, $A_h = \mathcal{H}_i^d[A_i]$, $L = \mathcal{H}_i^d[L_i]$, $N = \mathcal{H}_i^d[N_i]$, $E = \mathcal{H}_i^d[E_i]$, $\bar{M}_i = \mathcal{V}_j^d[M_{ji}]$, $\bar{M} = \mathcal{H}_i^d[\bar{M}_i]$, $\tilde{M}_j = \sum_k^d M_{jk}$, $\tilde{M} = \mathcal{V}_j[\tilde{M}_j]$, $M = \mathcal{V}_j^d[\mathcal{H}_i^d[M_{ji}]]$, $Q_\mu = blockdiag_i^d[(1-\mu_i)Q_i]$ and $\bar{h} = blockdiag_i^d[\bar{h}_i I_n]$.

Then the LPV time-delay system (2) is asymptotically stable and the $\mathcal{L}_2$ induced norm on the channel $w \to z$ is lower than γ.

Proof : The proof is given in appendix A. □

3. OBSERVERS SYNTHESIS

This section provides a sufficient condition to the existence of a gain-scheduled observer.

The state estimation $e(t)$ error dynamic expression

$$
\begin{aligned}
\dot{e}(t) &= A_{cl}(\rho)e(t) + A_{h_{cl}}(\rho)e_h(t)s + B_{cl}(\rho)w(t) \\
z_e(t) &= C_e(\rho)e(t)
\end{aligned}
\tag{7}
$$

with $A_{cl}(\rho) = A(\rho) - L(\rho)C_y(\rho)$, $A_{h_{cl}}(\rho) = A_h(\rho) - L(\rho)C_{yh}(\rho)$, $B_{cl}(\rho) = B(\rho) - L(\rho)E(\rho)$ and $C_e \in \mathbb{R}^{t\times n}$ is of full-row rank.

The aim of this part is to find observers which stabilize the estimation error dynamic and achieve an $\mathcal{H}_\infty$ attenuation property on the channel $w \to z_e$ lower than $\gamma > 0$.

Remark 3.1. It is worth noting that $C_e(\rho)$ is chosen then it would not depend on the parameter. Nevertheless, it would be interesting to create this dependence in order to obtain a varying precision on the state estimation with respect to parameter values. This 'virtual' parameter may be a system parameter or an exterior signal from an operator or a monitoring system. For example, with $0 < \rho < 1$, taking $C_e(\rho) = \begin{bmatrix} \alpha+\rho & 0 \\ 0 & \beta+1-\rho \end{bmatrix}$ with small α and β will associate good precision for the first component when ρ is near of 1 a detriment to the second component. When ρ is near of 0, the inverse happens.

The parameter dependent observer gain is first given in a rational dependence with respect to the parameters and it can be reduces to a simple polynomial dependence. The additive constraints relative to its construction are detailed in section 4.

Theorem 3.1. Consider system (7) with parameters and delays belonging to $\bigcup_{-1}^{1} \mathscr{P}^k, \mathscr{D}$. For given real scalars $\varepsilon_s, \varepsilon_u, \varepsilon_i$, if there exists matrices $Q_i, R_i \in \mathbb{S}^{n+}$, a continuously differentiable matrix valued function $P : U_\rho^1 \to \mathbb{S}^{n+}$, a scalar $\gamma > 0$ and continuous matrix functions $L_i, M_{ij}, Z, N_i : U_\rho \to \mathbb{R}^{n\times n}$, $E_i : U_\rho \to \mathbb{R}^{m\times n}$, $L_Z : U_\rho \to \mathbb{R}^{n\times s}$, $i, j = 1, \ldots, d$ such that the 2^p infinite dimensional LMIs (8) hold for all $\nu \in U_\nu$ and for all $\rho(t) \in U_\rho$.

$$
\Lambda(\nu, \rho) < 0 \tag{8}
$$

where $\Lambda = [\Lambda_{ij}]_{i,j}$, $j = 1\ldots, 6$, $\Lambda = \Lambda^T$ and

$$
\begin{aligned}
\Lambda_{11} &= \sum_i^d (Q_i + Sym(L_i(\rho))) - \varepsilon_s Sym(A'_{cl}(\rho)) \\
&\quad + \nu \frac{\partial P}{\partial \rho} \\
\Lambda_{21} &= -\mathcal{V}_i^d[\varepsilon_i A'_{cl}(\rho)] + \tilde{M}(\rho) - L(\rho)^T - \varepsilon_s A'_{h_{cl}}(\rho)^T \\
\Lambda_{22} &= -Q_\mu - Sym\left(M(\rho) + \mathcal{V}_i^d[\varepsilon_i A'_{h_{cl}}(\rho)]\right) \\
\Lambda_{31} &= P(\rho) + \sum_i^d N_i(\rho) - \varepsilon_u A'_{cl}(\rho) + \varepsilon_s Z(\rho)^T \\
\Lambda_{32} &= -N(\rho) - \varepsilon_u A'_{h_{cl}}(\rho) + \mathcal{H}_i^d[\varepsilon_i Z^T(\rho)] \\
\Lambda_{33} &= \sum_i^d \bar{h}_i R_i + \varepsilon_u Sym(Z(\rho)) \\
\Lambda_{41} &= \sum_i^d E_i(\rho) - \varepsilon_s B'_{cl}(\rho)^T \qquad \Lambda_{65} = 0_{n\cdot d\times q} \\
\Lambda_{42} &= -E(\rho) - \mathcal{H}_i^d[\varepsilon_i B'_{cl}(\rho)^T] \qquad \Lambda_{64} = \bar{h}E(\rho)^T \\
\Lambda_{43} &= \varepsilon_u B'_{cl}(\rho)^T \qquad \Lambda_{66} = -blockdiag_i^d[\bar{h}_i R_i]
\end{aligned}
$$

$$
\begin{aligned}
\Lambda_{44} &= -\gamma I_m & \Lambda_{51} &= C_e(\rho) & \Lambda_{52} &= 0_{t\times n\cdot d} \\
\Lambda_{53} &= 0_{t\times n} & \Lambda_{54} &= 0_{t\times m} & \Lambda_{55} &= -\gamma I_t \\
\Lambda_{61} &= \bar{h}L(\rho)^T & \Lambda_{62} &= \bar{h}\bar{M}(\rho)^T & \Lambda_{63} &= \bar{h}N(\rho)^T
\end{aligned}
$$

and $A_h = \mathcal{H}_i^d[A_i]$, $L = \mathcal{H}_i^d[L_i]$, $N = \mathcal{H}_i^d[N_i]$, $E = \mathcal{H}_i^d[E_i]$, $\bar{M}_i = \mathcal{V}_j^d[M_{ji}]$, $\bar{M} = \mathcal{H}_i^d[\bar{M}_i]$, $\tilde{M}_j = \sum_k^d M_{jk}$, $\tilde{M} = \mathcal{V}_j[\tilde{M}_j]$, $M = \mathcal{V}_j^d[\mathcal{H}_i^d[M_{ji}]]$, $Q_\mu = blockdiag_i^d[(1-\mu_i)Q_i]$, $\bar{h} = blockdiag_i^d[\bar{h}_i I_n]$, $A'_{cl}(\rho) = Z(\rho)A(\rho) + L_Z(\rho)C(\rho)$, $A'_{h_{cl}}(\rho) = Z(\rho)A_h(\rho) + L_Z(\rho)C_h(\rho)$ and $B'_{cl} = Z(\rho)B(\rho) + L_Z(\rho)E(\rho)$.

Then there exists a Luenberger's observer of the form (1) with gain $L(\rho) = Z^{-1}(\rho)L_Z(\rho)$ which asymptotically stabilizes the state estimation error with an $\mathcal{L}_2$ induced norm on channel $w \to z_e$ lower than γ.

Proof : The proof is given in appendix B. $\square$

4. DISCUSSION ON MATRIX FUNCTIONS COUPLING

As presented in theorem 3.1, the choice of basis functions and the number of gridding points are crucial in order to not to raise conservatism. Moreover a coupling between the two matrix functions $Z(\rho)$ and $L(\rho)$ appears and may be troublesome and needs particular attention.

4.1 Number of gridding points and choice of basis functions

There does not exist any method to know a priori how much gridding points to consider. If 0 belongs to the set of parameters or if there exists parameters for which the system matrices radically changes (become unstable...), these values must be considered explicitly. The other gridding points only cover the whole parameter space. The number must not be too large to reduce computational complexity but not be too small to well characterize the system. Moreover, the space between points may be variable: it should be small whenever small parameters variations induces large perturbations on matrices and vice-versa.

The choice of type and the number of basis function is considered to relax infinite dimensional matrices while projecting them on a basis $X(\rho) \simeq \sum f_i(\rho)X_i$. This is quite difficult because no theory exists on how to make these choices. Generally, polynomials are used for ease of simplicity.

4.2 Matrices coupling

If Z (non symmetric) does not depend on parameters, in that case, the matrix Z does not need a particular inertia except that it must be nonsingular. If we obtain a singular matrix, a way to overcome the problem is to perturb it until obtain a nonsingular matrices in order to compute the observer gain.

Nevertheless, if $Z(\rho)$ depends on the parameter, it is rather more difficult because $Z(\rho)$ must be invertible over the whole parameter space. In (Bhatia, 1997, p.254)), some discussions are provided on matrix functions. The main problem is the zero crossing of theses functions which induces the singularity of the matrix for one parameter configuration. A simple way to avoid this problem is to fix the matrix as symmetric and force it to a certain definiteness (positive or negative). This introduces an additional conservatism but actually simplifies the problem. Moreover, with a good choice of basis functions it is possible to insure the property over the whole parameter space only considering the parameter polytope vertices and some basic assumptions on decision matrices. For example, with $P(\rho) = P_0+P_1\rho+P_2\rho^2$ with $P_2 \geq 0$ then $P(\rho) < 0$ on U_ρ if and only if $P(\rho^\circ) < 0$ for all $\rho^\circ \in \partial U_\rho$ where ∂U_ρ denote the boundary of the set U_ρ. Note this set is finite because it only involves union of compact sets on the real line.

The matrix non-singularity can also be given in term of a rank constraint over the parameter space but this does not simplify the problem because this leads to an infinite dimensional non-convex constraint.

It is possible to give a simple algorithm to find the structure of matrices P and Z. We assume here that we have a bound on the degree of Z and L_Z defining a constraint of the controller complexity. Note that the complexity of P does not implies the complexity of the controller since they are uncoupled. Hence, a complex P will generally leads to accurate results in terms of $\mathcal{H}_\infty$ norm and delay stability margin. The complexity of the controller is directly deduced from the complexity of $Z(\rho)$ and $L_Z(\rho)$ since $L(\rho) = Z^{-1}(\rho)L_Z(\rho)$. Note that the controller is rational with respect to the parameters if and only if $Z(\rho)$ depends on the parameters.

As the matrix P can only depend on smooth parameters and it would be interesting to include nonsmooth parameters manually in the controller when choosing its structure.

5. EXAMPLE

Consider system (2) with matrices

$$A = \begin{bmatrix} -5+\rho & 0 \\ 1 & -3+\rho \end{bmatrix} \; A_h = \begin{bmatrix} 0.1 & 0 \\ -0.1+\rho & -0.1 \end{bmatrix}$$
$$E = \begin{bmatrix} 1 & 1 \end{bmatrix} \qquad \mu \leq 0.7$$

$$B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \; C_y = \begin{bmatrix} 1 & 2+\rho \end{bmatrix} \; C_{yh} = \begin{bmatrix} 5 & 1 \end{bmatrix}$$

Using theorem 3.1 with $\varepsilon_s = \varepsilon_1 = \varepsilon_u = 1$, $\rho(t) \in [-2, 2]$ with a 11 points grid, $h_{max} = 0.2$ and constant matrix functions leads to a γ upper-bound of 0.6988 with observer gain $L = \begin{bmatrix} -0.1856 & 0.7390 \end{bmatrix}^T$. Now we consider, constant decision matrices and a quadratic dependent observer gain with a quadratic dependence with respect to the parameters (ie. $L_Z(\rho) = L_{Z_0} + \rho L_{Z_1} + \rho^2 L_{Z_2}$) leads to a γ upper-bound of 0.5331 and an observer gain of $L(\rho) = Z^{-1}L_Z(\rho)$ with

$$Z = \begin{bmatrix} -0.3698 & 0.0535 \\ 0.0551 & -0.1939 \end{bmatrix} \qquad rank(Z) = 2$$

$$L_Z(\rho) = \begin{bmatrix} 0.0059\rho^2 - 0.0375\rho + 0.0707 \\ 0.0050\rho^2 - 0.0268\rho - 0.0664 \end{bmatrix}$$

The explicit value of the parameter dependent observer gain is given by

$$L(\rho) = \begin{bmatrix} -0.0206\rho^2 + 0.1265\rho - 0.1477 \\ -0.0314\rho^2 + 0.1740\rho + 0.3005 \end{bmatrix}$$

For $\rho(t) = 2sin(3t + \pi/2)$, $h(t) = 0.09sin(6t) + 0.2$ and step exogenous inputs, we obtain the simulation results presented in figures 1 and 2.

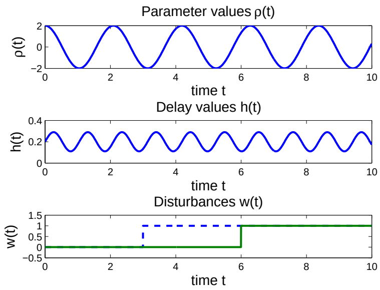

Fig. 1. Parameter, delay and disturbances

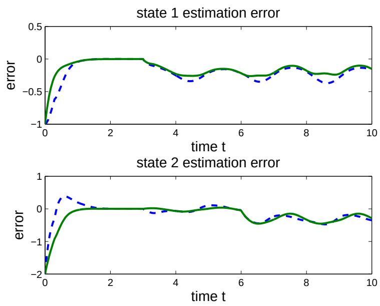

Fig. 2. Estimation errors for both constant and parameter dependent observer gain

On figure 2, the plain lines represents estimation errors obtained with parameter dependent observer gain and the dashed lines represents those obtained with the constant observer gain. The error converges faster to zero while using a parameter dependent observer. Moreover, the disturbances are more attenuated ($\mathcal{H}_\infty$ disturbance attenuation is smaller) considering also an parameter dependent observer. Note that this is obtained for the a particular triplet $(\varepsilon_s, \varepsilon_1, \varepsilon_u)$ and thus our result provided above are certainly suboptimal.

6. CONCLUSION

We have developed in this paper a method to synthesize gain scheduled observers for parameter varying time delay systems. The results are presented through gridded infinite dimensional LMIs. This method is based on adding several supplementary degrees of freedom in the Lyapunov-Krasovskii functional derivative. The benefits of this approach is demonstrated through an example. A future paper will compare this approach to another using the Jensen's inequality rather than the Newton's formula which leads to better results.

Appendix A. PROOF OF THEOREM 2.1

Consider the Lyapunov-Krasovskii functional:

$$\begin{aligned} V(x,t) = x^T P(\rho) x(t) + \sum_i^d \int_{t-h_i(t)}^t x(\theta)^T Q_i x(\theta) d\theta \\ + \sum_i^d \int_{-\bar{h}_i}^0 \int_{t+\theta}^t \dot{x}^T(s) R_i \dot{x}(s) ds d\theta \end{aligned} \tag{A.1}$$

Consider also the supply function $s(w, z)$ defined as

$$s(w,z) = \gamma w^T(t) w(t) - \gamma^{-1} z^T(t) z(t) \tag{A.2}$$

which describes the $\mathcal{H}_\infty$ input/output constraint.

Then remark using Newton's formula this equality holds (we drop here the dependence on ρ for ease of clarity) for some matrices L_i, M_{ji}, N_i and E_i $(i, j \in \{1, \dots, d\})$ of appropriate dimensions:

$$\begin{aligned} 2\sum_i^d \Bigg(\Big[x^T(t) L_i + x_h^T(t) \underset{j}{\mathcal{V}}[M_{ji}] + \dot{x}^T(t) N_i + w^T(t) E_i \Big] \\ * \left[x(t) - x(t - h_i(t)) - \int_{t-h_i(t)}^t \dot{x}(s) \right] \Bigg) = 0 \end{aligned} \tag{A.3}$$

Note using the system expression (2) we have also

$$\begin{aligned} 2\left[x^T(t) S + x_h(t)^T \underset{i}{\mathcal{V}}[T_i] + \dot{x}^T(t) U \right] \\ * \left[\dot{x}(t) - Ax(t) - A_h x_h(t) - Bw(t) \right] = 0 \end{aligned} \tag{A.4}$$

for any matrices S, T_i, U $(i \in \{1, \dots, d\})$ and

$$\bar{h}_i \chi(t)^T X_i \chi(t) - \int_{t-h_i(t)}^t \chi(t)^T X_i \chi(t) ds \ \geq \ 0 \tag{A.5}$$

for $X_i = X_i^T \geq 0$, $i \in \{1, \dots, d\}$ and $\chi(t) = col\,(x(t), x_h(t), \dot{x}(t), w(t))$.

Then computing the derivative of (A.1) along the trajectories solutions of the system (2) leads for all $\nu \in U_\nu$ to

$$\dot{V} = \dot{x}^T P(\rho)x(t) + x(t)^T P(\rho)x(t) + \frac{\partial P(\rho)}{\partial \rho}\dot{\rho}$$

$$+\sum_i^d [x(t)^T Q_i x(t) - (1-\dot{h}_i(t))x(t-h_i(t))^T Q_i(\star)^T$$

$$+\bar{h}_i\dot{x}(t)^T R_i \dot{x}(t)] - \sum_i \int_{t-h_i(t)}^t \dot{x}^T(s)R_i\dot{x}(s)ds < 0$$

Bound $\dot{h}_i(t)$ by μ_i and we must relax the term $\dot{\rho}$ using a convex argument. Thus we consider $\dot{\rho}$ belonging to a polytope whose vertices set is defined by U_ν (which is introduced in (5)) leads to:

$$\dot{V} \leq \dot{x}^T P(\rho)x(t) + x(t)^T P(\rho)x(t) + \frac{\partial P(\rho)}{\partial \rho}\nu$$

$$+\sum_i^d [x(t)^T Q_i x(t) - (1-\mu_i)x(t-h_i(t))^T Q_i x(t-h_i(t))$$

$$+\bar{h}_i\dot{x}(t)^T R_i \dot{x}(t)] - \sum_i \int_{t-h_i(t)}^t \dot{x}^T(s)R_i\dot{x}(s)ds < 0$$

Then we add the constraint (A.2), expand it considering the system expression (2) and perform a Schur complement on terms $\star^T\gamma^{-1}\star$. Finally add constraints (A.3), (A.4) and (A.5). That leads to the inequality:

$$\dot{V} \leq \chi^T(t)\left(\Phi + \sum_i \bar{h}_i X_i\right)\chi(t)$$
$$- \sum_i \int_{t-h_i(t)}^t \zeta^T(t,s)\Psi_i\zeta(t,s)ds < 0$$

with $\zeta(t,s) = col(\chi(t), \dot{x}(s))$ and for $i,j \in \{1,\dots,5\}$ we have $\Phi = [\Lambda_{ij}]$ and

$$\Psi_i = \begin{bmatrix} X_i & \star \\ \kappa_i^T & R_i \end{bmatrix}, \ \kappa_i^T = \begin{bmatrix} L_i^T & \bar{M}_i^T & N_i^T & E_i^T \end{bmatrix}$$

where the Λ_{ij} are defined in theorem 2.1.

If $\Psi_i \geq 0$, we must only check the negative definiteness of $\Phi + \sum_i \bar{h}_i X_i$. Taking $X_i = \kappa_i R_i^{-1}\kappa_i^T$ implies $\Psi \geq 0$ and we inject their values into the sum $\Phi + \sum \bar{h}_i X_i$. Perform a Schur's complement wrt. to terms $\star^T R_i^{-1}\star$ and we obtain the LMI provided by theorem 2.1.

Appendix B. PROOF OF THEOREM 3.1

Compute the state estimation error 7, inject it in the LMIs of theorem 2.1. This leads to a BMI due to terms T_iL, SL and UL. Then posing $T_i = \varepsilon_i Z$, $U = \varepsilon_u Z$ and $S = \varepsilon_s Z$ for some non-zero scalars ε_i, ε_u and ε_s leads to the LMIs of theorem 3.1.

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